04.03.2015 **Solution**

Advanced Functional Analysis - Mid-Sem Exam - Semester II

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1. **Question:** Let X be a topological vector space over the field $\mathbb{C}$. Suppose $f : X \mapsto \mathbb{C}$ is a linear functional. Show that f is continuous iff $ker(f)$ is closed in X.

   Solution: See Rudin Functional analysis book Th 1.18 and page no 15.

   $\square$

2. **Question:** Suppose K is a compact subset of a topological vector space Z. Show that $K - K$ is compact.

   Solution: Consider the map $\Phi : Z \times Z \mapsto Z$ defined as $\Phi(x, y) = x - y$ $\square$

3. **Question:** Let X be a separable Banach space with countable dense set $\{x_n : n \geq 0\}$. For $n \geq 1$, let f_n be a linear functional on X satisfying $f_n(x_n) = \|x_n\|$, and $\|f_n\| = 1$. Define $T : X \mapsto l^\infty$ by $T(x) = (f_1(x), f_2(x), ...,)$. Show that T is an isometry.

   Solution: Note that $|f_i(x)| \leq \|f_i\|\|x\|$ for all $x \in X$. It follows that $\|T(x)\| \leq \|x\|$ for all $x \in X$. Observe that

$$\begin{aligned} \|T(x_n)\| &= sup\{|f_i(x_n)| : i \in \mathbb{N}\} \\ &\geq |f_n(x_n)| \\ &= \|x_n\| \end{aligned}$$

   Therefore $\|T(x_n)\| = \|x_n\| \ \forall \ n \in \mathbb{N}$. By the conitinuity of T and the denseness of $\{x_n : n \in \mathbb{N}\}$ it follows that $\|T(x)\| = \|x\| \ \forall \ x \in X$.

   $\square$

4. **Question:** Let X, Y be normed linear spaces and let X be finite dimensional. Suppose $T : X \mapsto Y$ is a linear map. Show that T is continuous.

   Solution: Let $e_1, e_2, ..., e_n$ be a basis for X. Then it follows that $\|T(x)\| \leq \alpha\|x\|_1$ where $\alpha = max\{\|T(e_1)\|, \|T(e_2)\|, ..., \|T(e_n)\|\}$. $\square$

5. **Question:** Let X, Y be Banach spaces. Let $T : X \mapsto Y$ be a bounded linear map and T is onto. Show that T is an open map.

   Solution: Open mapping theorem. $\square$

6. **Question:** Let V be a topological vector space. A non-empty subset A of V is said to be absorbing if for each $x \in V$, there exists a $t > 0$ such that $\frac{x}{t}$ is in A. Show that every open neighborhood of 0 is absorbing. Show that every $y \neq 0$ has an open neighborhood which is not absorbing.

   Solution: Let A be a nbhd of 0 and $x \in V$, then there exists a $\delta > 0$ and some nbhd B of x such that $\beta B \subset A$ whenever $|\beta| < \delta$. There exists a $t_0 > 0$ such that $\frac{1}{t_0} < \delta$ and for this t_0 we have $\frac{x}{t_0} \in A$. For $y \neq 0$ there exists a nbhd A which does not contain 0. There exists no $t > 0$ such that $\frac{0}{t} \in A$. This proves that A is not absorbing. $\square$

7. **Question:** Let $C[0,1]$ be the Banach space of complex valued continuous functions on the interval, with the supremum norm. Let $C_0[0.1]$ be the subspace: $C_0[0,1] = \{f \in C[0,1] : f(0) = 0\}$. Identify the set of extreme points of closed unnit balls of $C[0,1]$ and $C_0[0.1]$.

   Solution:

   f is an extreme point if and only if $|f(x)| = 1 \ \forall x \in [0,1]$. Unit ball of $C_0[0,1]$ has no extreme points.

   $\square$